TOPIC: CONVERGENCE TESTS

Divergence Test (n[th]-Term Test)

◆ The divergence test proves that an infinite series *diverges* by taking the limit of a_n as $n \to \infty$.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
Divergence Test	$\sum_{n=1}^{\infty} a_n$	Does NOT determine convergence	If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ DNE	If $\lim_{n\to\infty} a_n = 0$, test is inconclusive

EXAMPLE Use the divergence test to determine if the following series diverge.

$(\boldsymbol{A})$

$$\sum_{n=1}^{\infty} \frac{4^n}{n^4}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{3n}{n^2+1}$$

$(\boldsymbol{C})$

$$\sum_{n=1}^{\infty} \frac{2n!}{3n!-4}$$

$(\boldsymbol{D})$

$$\sum_{n=1}^{\infty} (-1)^n$$

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PRACTICE Use the divergence test to determine if the following series diverge or state that the test is inconclusive.

(A)

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$$

(B)

$$\sum_{n=1}^{\infty} \frac{10^n}{n!}$$

(C)

$$\sum_{n=1}^{\infty} \frac{n^2}{n(n^2 - 1000)}$$

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Integral Test

◆ The integral test determines convergence of infinite series using ___________ integrals.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
Integral Test	$\sum_{n=N}^{\infty} a_n$ $a_n =$ ______ must be: ▸ Positive ▸ Continuous ▸ Decreasing for $x \geq N$	The integral $\int_{\_\_\_}^{\_\_\_} \_\_\_\_ \, dx$ converges.	The integral $\int_{\_\_\_}^{\_\_\_} \_\_\_\_ \, dx$ diverges.	Use when a_n can be easily integrated.

◆ An improper integral converges if the limit exists (a finite number) and diverges if the limit DNE.

EXAMPLE Use the integral test to determine if the following series converges.

$$\sum_{n=0}^{\infty} e^{-n}$$

$f(x)$ for $x \geq$ ___ is:

Positive ☐

Continuous ☐

Decreasing ☐

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PRACTICE Explain why the integral test does not apply to the series.

(A)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$$

(B)

$$\sum_{n=4}^{\infty} \frac{1}{n + \sin n}$$

(C)

$$\sum_{n=0}^{\infty} \frac{n}{n^2 - 1}$$

PRACTICE Confirm that the integral test applies and then use the integral test to determine convergence of the series.

(A)

$$\sum_{n=1}^{\infty} \frac{\arctan n}{n^2 + 1}$$

(B)

$$\sum_{n=1}^{\infty} \frac{1}{3n + 2}$$

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P-Series and Harmonic Series

◆ We can quickly determine convergence by recognizing that a series is a p-series or a harmonic series.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	p $\quad$ 1	p $\quad$ 1	
Harmonic Series	$\sum_{n=1}^{\infty} \frac{1}{n}$	Does _____ converge	___________ diverges	A general harmonic series $\sum_{n=1}^{\infty} \frac{1}{an+b}$ also diverges.

EXAMPLE Determine whether the given series are convergent.

(A) $\sum_{n=1}^{\infty} \frac{1}{n^{7/5}}$

(B) $\sum_{n=1}^{\infty} \frac{1}{n-1}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

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PRACTICE Determine whether the given series are convergent.

(***A***)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^3}}$$

(***B***)

$$\sum_{n=1}^{\infty} \frac{5}{3n-7}$$

(***C***)

$$\sum_{n=1}^{\infty} \frac{n^e}{n^2}$$

(***D***)

$$\sum_{n=1}^{\infty} n^{-1} + n^{-3}$$

TOPIC: CONVERGENCE TESTS

Direct Comparison Test

◆ Direct Comparison Test (DCT) uses convergence of known series b_n to find convergence of series in question a_n.

► Use DCT when divergence and integral tests are insufficient & series is not a "special" series.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ $a_n, b_n >$ ___	If $\quad\leq\quad$ and $\sum b_n$ (larger) converges, $\sum a_n$ (smaller) converges.	If $\quad\leq\quad$ and $\sum b_n$ (smaller) diverges, $\sum a_n$ (larger) diverges.	For b_n, we often use a special series (geometric, p-series, harmonic).

EXAMPLE Determine whether the given series are convergent.

(A)

$$\sum_{n=1}^{\infty} \frac{\ln n}{2n+1}$$

(B)

$$\sum_{n=1}^{\infty} \frac{5}{n^3+4}$$

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PRACTICE Use the Direct Comparison Test to determine whether each series converges.

$(\boldsymbol{A})$

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}-2}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$$

Hint: Compare to $b_n = \frac{1}{n^2}$

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Limit Comparison Test

◆ Limit Comparison Test uses the limit of the ratio of given series a_n to chosen series b_n to determine convergence.

► Use Limit Comparison Test if given a more complicated series that can't be *directly* compared to a known series.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ $a_n,\ b_n >$___ $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$	$L \quad 0$ and $\sum b_n$ converges.	$L \quad 0$ OR $L \to$ ____ and $\sum b_n$ diverges.	Most useful when b_n isn't clearly: • a _________ series • $> a_n$ or $< a_n$

EXAMPLE Determine whether the given series is convergent using the Limit Comparison Test.

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{5n^3 - 1}$$

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PRACTICE Use the Limit Comparison Test to determine whether each series converges.

$(\boldsymbol{A})$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+3}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{n}{(n-1)3^{n+1}}$$

PRACTICE Use the Limit Comparison Test to determine if the following series converges.

$(\boldsymbol{A})$

$$\sum_{n=1}^{\infty} \frac{1}{2n^2 - n\cos(n\pi)}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n}$$

TOPIC: CONVERGENCE TESTS

Alternating Series Test

◆ An alternating series has terms that alternate in ________ & often contains: $(-1)^n$, $(-1)^{n\pm1}$, $\cos n\pi$, or $\sin\frac{n\pi}{2}$.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
Alternating Series Test	$\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$	Two conditions are met: • $\lim_{n\to\infty}(a_n) =$ • $0 <$ $\leq a_n$ for all n (decreasing)	One of the conditions is **NOT** met	If $\sum a_n$ converges, then it is… • **Absolutely** convergent if $\sum \lvert a_n \rvert$ ____verges. • **Conditionally** convergent if $\sum \lvert a_n \rvert$ ____verges.

EXAMPLE Determine whether the series converges absolutely, converges conditionally, or diverges.

(A) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+3}$

(B) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n!}{3^n}\right)$

(C) $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$

◆ Absolute convergence ($\sum \lvert a_n \rvert$ converges) implies convergence of $\sum a_n$.

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PRACTICE Use the Alternating Series Test to determine if the following series are conditionally convergent, absolutely convergent, or divergent.

(A)

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{\ln n}$$

(B)

$$\sum_{k=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^3}$$

(C)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{3}{4n+5}\right)$$

TOPIC: CONVERGENCE TESTS

Alternating Series Remainder

◆ Alternating series remainder measures how accurately the nth partial sum S_n estimates the *actual* sum S.

► The error (remainder) R_n is given by __________ and is *no greater than* the first __________ term a_{n+1}.

Alternating Series Remainder

If the **alternating series**

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converges by the alternating series test, then

$$|R_n| = |S - S_n| \qquad a_{n+1}$$

EXAMPLE

Consider the convergent series: $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^3}\right)$

($\boldsymbol{A}$) Use the remainder estimate $|R_n| \leq a_{n+1}$ to determine a bound on the error R_8.

($\boldsymbol{B}$) Determine how many terms should be used to estimate the entire series with an error less than 10^{-3}.

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PRACTICE

Use the Alternating Series Test to $(\boldsymbol{A})$ determine the convergence or divergence of the series and $(\boldsymbol{B})$ approximate the sum of the series using the first five terms.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}+2}$$

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Ratio Test

- The Ratio Test uses the limit of the ratio of a_{n+1} and a_n to determine convergence.
  - Ratio Test is often used for series that contain factorials, exponentials, and/or powers.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
Ratio Test	$\sum_{n=1}^{\infty} a_n$ Let $\lim_{n\to\infty} \left\lvert \frac{\quad}{\quad} \right\rvert = r$	r ____ 1	r ____ 1	Inconclusive if r ____ 1, use another convergence test

EXAMPLE Determine whether the given series is convergent using the Ratio Test.

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

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EXAMPLE Determine whether the given series are convergent using the Ratio Test.

$(\boldsymbol{A})$

$$\sum_{k=1}^{\infty} \frac{2^k}{(k+1)!}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$$

PRACTICE Determine whether the given series are convergent using the Ratio Test.

$(\boldsymbol{A})$

$$\sum_{n=1}^{\infty} \frac{8^n}{(2n)!}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{n^5}{(6)^n}$$

$(\boldsymbol{C})$

$$\sum_{n=1}^{\infty} \frac{n!}{3^n}$$

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Root Test

◆ The Root Test uses the limit of the nth root of the series to determine convergence.

► Root Test is often used when a_n is raised to the nth power.

CONVERGENCE TESTS				
Name	**Series**	**Converges if…**	**Diverges if…**	**Additional Info**
Root Test	$\sum_{n=1}^{\infty} (a_n)^n$ Let $\lim_{n\to\infty} \sqrt[(\quad)]{\lvert \quad \rvert} = r$	r _____ 1	r _____ 1	Inconclusive if r _____ 1, use another convergence test

EXAMPLE Determine whether the given series are convergent using the root test.

(A) $\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n+1))^n}$

(B) $\sum_{n=1}^{\infty} \frac{(2n^3-3)^n}{(4n^3+n)^n}$

TOPIC: CONVERGENCE TESTS

PRACTICE Determine if the following series converges, diverges, or is inconclusive.

$(\boldsymbol{A})$

$$\sum_{n=1}^{\infty}\left(\frac{4n+1}{n-2}\right)^{n}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty}\left(\frac{2n^2-1}{n^2+5}\right)^{-n}$$

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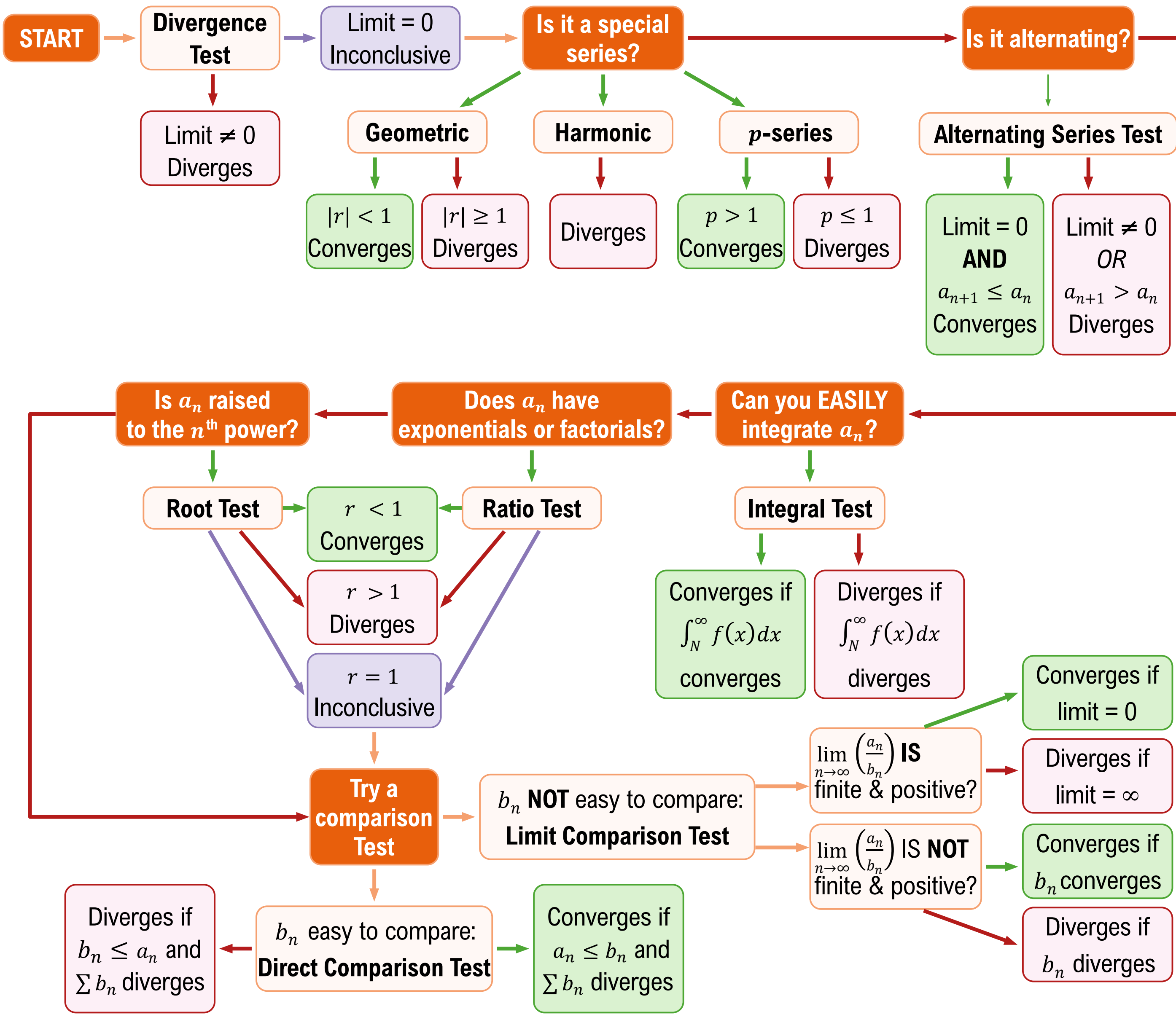
Choosing a Convergence Test



EXAMPLE Choose a convergence test for each of the following series.

(A)

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^4}$$

(B)

$$\sum_{n=1}^{\infty} \frac{5}{n^4 + 2n^2 - 24}$$

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PRACTICE Determine the convergence or divergence of the series.

$(\boldsymbol{A})$

$$\sum_{k=1}^{\infty} \frac{(-1)^k 4}{3k+2}$$

$(\boldsymbol{B})$

$$\sum_{n=1}^{\infty} \frac{5^{n-1}}{2^n}$$

$(\boldsymbol{C})$

$$\sum_{n=1}^{\infty} \frac{n \cdot 8^n}{(n+1)!}$$

$(\boldsymbol{D})$

$$\sum_{k=1}^{\infty} \frac{7k^2}{5k+3}$$