TOPIC: TAYLOR SERIES & TAYLOR POLYNOMIALS

Taylor Series

◆ Recall: A power series is a polynomial w/ infinite terms represented by

Recall
$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

► A Taylor Series is a type of power series where c_n is determined using the _______________ of a function.

EXAMPLE Find the Taylor series of $f(x) = \ln x$ centered at $a = 1$.

New **Taylor Series**

$$\sum_{n=0}^{\infty} \frac{\quad}{\quad}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots$$

(Taylor Series centered at $x = a$)

Derivatives of f	Derivatives of f at $x = 1$
$f = \ln x$	$f(1) = \ln \quad =$
$f' =$	$f'(1) =$
$f'' =$	$f''(1) =$
$f''' =$	$f'''(1) =$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\quad)}{n!}(x- \quad)^n$$

◆ *Maclaurin series* is a special case of Taylor series that is centered at ____:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

(Maclaurin Series)

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PRACTICE

Find the Taylor Series of $f(x) = \cos x$ centered at $x = \pi$. Then, write the power series using summation notation.

Derivatives of f	Derivatives of f at $x = \pi$
$f =$	$f(\pi) =$
$f' =$	$f'(\pi) =$
$f'' =$	$f''(\pi) =$
$f''' =$	$f'''(\pi) =$
$f^{(4)} =$	$f^{(4)}(\pi) =$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\quad)}{n!}(x - \quad)^n$$

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EXAMPLE Answer the following questions.

(***A***) Find the Maclaurin Series of $f(x) = e^x$. Then, write the power series using summation notation.

Derivatives of $\boldsymbol{f}$	Derivatives of $\boldsymbol{f}$ at $\boldsymbol{x = 0}$
$f =$	$f(0) =$
$f' =$	$f'(0) =$
$f'' =$	$f''(0) =$
$f''' =$	$f'''(0) =$
$f^{(4)} =$	$f^{(4)}(0) =$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\quad)}{n!}(x - \quad)^n$$

(***B***) Use the Maclaurin series for $f(x) = e^x$ to find the Maclaurin series for $f(x) = e^{2x}$.

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Convergence of Taylor and Maclaurin Series

◆ Taylor & Maclaurin series have **intervals of convergence** which are found in the same way as power series.

EXAMPLE Determine the convergence of the Taylor series for $f(x) = \ln x$ centered at $x = 1$.

Write the general term: $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n =$

$$= 0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

Use convergence test:

Recall

Ratio: $\lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| < 1$

Root: $\lim_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} < 1$

Geometric: $\sum ar^n; |r| < 1$

Determine endpoint convergence:

$x =$ ___: $\sum_{n=0}^{\infty} (-1)^n \frac{(\quad -1)^{n+1}}{n+1}$

$x =$ ___: $\sum_{n=0}^{\infty} (-1)^n \frac{(\quad -1)^{n+1}}{n+1}$

Write interval of convergence:

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PRACTICE Find the interval of convergence for the Taylor series for $f(x) = \sin x$ centered at $x = \frac{\pi}{2}$.

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PRACTICE Find the interval of convergence for the Maclaurin series for $f(x) = \tan^{-1} x$.

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Taylor Polynomials

Recall
$s_n = a_1 + a_2 + a_3 + \cdots + a_n$

- A Taylor polynomial $p_n(x)$ of degree n is a ____________ of a Taylor series & is used for approximating functions.
  - Taylor polynomials centered at $x = 0$ are also called Maclaurin polynomials.

EXAMPLE Find the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = e^x$.

New — **n^{th} Taylor Polynomial**

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Taylor polynomial centered at $x = a$)

$p_0(x) =$

$p_1(x) =$

$p_2(x) =$

$p_3(x) =$

EXAMPLE Approximate $e^{0.2}$ to four decimal places using the third-degree Maclaurin polynomial for $f(x) = e^x$.

$p_3(x) =$

- Using higher-________ Taylor polynomials gives more accurate approximations.

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Recall

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Taylor polynomial centered at $x = a$)

PRACTICE Answer the following questions.

($\boldsymbol{A}$) Find the Taylor polynomials of order 0, 1, 2, and 3 for $f(x) = \ln(x)$ centered at $x = 1$.

($\boldsymbol{B}$) Approximate $\ln 1.5$ to four decimal places using the third-degree Taylor polynomial for $f(x) = \ln x$.

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Recall

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Taylor polynomial centered at $x = a$)

PRACTICE Answer the following questions.

(***A***) Find the Maclaurin polynomials of order 0, 1, 2, and 3 for $f(x) = \sin x$.

(***B***) Approximate $\sin 0.3$ to four decimal places using the third-degree Maclaurin polynomial for $f(x) = \sin x$.

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Topic Resource: Taylor Series of Common Functions

Taylor Series (centered at $x = 0$)	Interval of Convergence
$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1,1)$
$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$	$(-1,1)$
$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^n}{n}$	$(-1,1]$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$[-1,1]$